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The free energy of $N = 4$ super Yang-Mills and the AdS/CFT correspondence

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Abstract: We compute the high-temperature limit of the free energy for four-dimensional $N = 4$ supersymmetric SU(N_c) Yang-Mills theory. At weak coupling we do so for a general ultrastatic background spacetime, and in the presence of slowly-varying background gauge fields. Using Maldacena's conjectured duality, we calculate the strong-coupling large-N_c expression for the special case that the three-space has constant curvature. We compare the two results paying particular attention to curvature corrections to the leading order expressions.

Keywords: Strong Coupling Expansion, Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories, Extended Supersymmetry.

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1. Introduction

There is substantial evidence to support Maldacena's conjectured duality relating superstring theory in anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) [1, 2, 3] — for a comprehensive review, see ref. [4]. In particular, type IIB superstring theory on $AdS_5 \times S^5$ is dual to four-dimensional $N = 4$ super-Yang-Mills (SYM) theory with gauge group SU(N_c) [1]. An interesting aspect of this duality is the study of the behavior of the SYM theory at finite temperature [1, 3, 5, 6, 7, 8, 9, 10, 11]. In the superstring theory for sufficiently high temperatures, the thermal state is described by an asymptotically AdS black hole [1, 3]. One finds that, at a qualitative level, one recovers many of the expected physical properties of the thermal SYM system from the black hole geometry. A quantitative comparison is more difficult because supergravity provides a description of the SYM theory at strong coupling, and so no direct calculations can be made reliably in the SYM theory. In some cases [6, 12], e.g., the free energy density, perturbative calculations at weak coupling still reproduce the strong coupling results up to factors of order one. So it would appear that the corresponding coefficients are smooth functions of the effective coupling which interpolate between the strong and weak coupling results. In the case of the free energy density, subleading corrections have been calculated at both strong [9] and weak [10] coupling, and the results are suggestive that the interpolation may even be achieved by simply a monotonic function.

Whether or not there is a smooth interpolation between strong and weak coupling is a question deserving close scrutiny. By carefully examining the expansions of the free energy at small and large coupling, and by making some simple assumptions about the behavior of the expansion coefficients, Li [11] has argued that it is impossible to smoothly interpolate between these two coupling regimes of the SYM theory. Hence Li concluded that there must be a phase transition at some critical coupling when the SYM theory is at any finite temperature. The primary motivation for the present paper was to investigate the free energy calculations in more detail to look for evidence of such a phase transition. As well as introducing a finite temperature, we consider the SYM theory on a curved background space. Finite temperature is included by working with a four-dimensional, ultrastatic euclidean spacetime for which the euclidean 'time' direction is periodically identified with period $\beta = 1/T$. For the spatial geometry, we pay particular attention to three-geometries of constant curvature κ/ℓ^2 (with $\kappa = 0, \pm 1$) since these are the cases for which we may also compute the strong-coupling free energy using the AdS/CFT correspondence. When comparing the results obtained for weak and strong couplings, we follow how they depend on the radii, β and ℓ, in the limit $\beta/\ell \ll 1$ (which corresponds to the high-temperature limit).

We organize the paper as follows. Section 2 states the preliminaries, describing the $N = 4$ super-Yang-Mills theory. Section 3 then addresses the weak-field calculation, which may be performed quite generally, using well-known heat-kernel techniques. We compute the weak coupling effective action for SYM at finite temperature, and in the presence of slowly-varying background gauge and gravitational fields. The strong-coupling string/supergravity calculation is then given in section 4. A brief discussion of our results is given in section 5.

2. $N = 4$ super-Yang-Mills theory

The field content of $N = 4$ SYM theory [13] is $\{A^a_\mu, \lambda^a_i, \varphi^a_r\}$, where $a = 1, \ldots, d_G$ runs over the adjoint representation of the gauge group, G, $i = 1, \ldots, 4$ counts the spin-half fields, and $r = 1, \ldots, 6$ labels the scalars. Where necessary we will choose $G = \mathrm{SU}(N_c)$, for which $d_G = N_c^2 - 1$, although this choice does not play an important role in the weak coupling calculations.

The action of the theory may be conveniently formulated in terms of $N = 1$ superfields, of which we require one gauge multiplet, $\mathcal{W}^a_L$, and three matter multiplets, ϕ^a_m, all in the adjoint representation. The action is then given by minimal kinetic terms for all fields, gauge interactions, plus those interactions derived from the $N = 1$ superpotential:

$$W = \frac{i\sqrt{2}}{3!}\, \epsilon^{mnp}\, c_{abc}\, \phi^a_m\, \phi^b_n\, \phi^c_p\,. \tag{2.1}$$

Here c_{abc} represent the completely antisymmetric structure constants for the gauge group.

This superpotential is manifestly invariant under a global SU(3) flavour symmetry (acting on the indices m, n and p), as well as a U(1) R-symmetry for which the charge assignments of the fields are $R(\mathcal{W}^a_L) = 1$ and $R(\phi^a_r) = 2/3$. These are subgroups of a larger SU(4) flavour symmetry which this model enjoys, defined as the automorphism of the supersymmetry algebra which rotates the four supersymmetry generators amongst themselves. Only the $\mathrm{SU}(3) \times U_R(1)$ subgroup is manifest when the theory is expressed in terms of $N = 1$ superfields.

The theory is also scale invariant, even at the quantum level [14]. This may be argued within perturbation theory using the nonrenormalization theorems of $N = 2$ supersymmetry, or by constructing the most general $N = 1$ supersymmetric action with this field content which admits the above-mentioned SU(4) symmetry.

The theory's scalar potential is

$$V = \sum_{am} \left| \frac{\partial W}{\partial \phi^a_m} \right|^2 , \tag{2.2}$$

whose minima are described by the supersymmetry-preserving flat directions for which $c_{abc}\, \phi^b_n \, \phi^c_p = 0$. Semiclassically, nonzero fields along these flat directions cost no energy and spontaneously break the gauge group, the global SU(4) symmetry and scale invariance. A number of Goldstone and massless gauge multiplets therefore figure prominently among the low-energy states, at least at weak coupling.

In the following, we will examine the free energy of this theory at finite temperature and in curved background spacetimes. A nontrivial result arises because both the temperature and background curvatures typically break supersymmetry, thereby permitting a nonvanishing free energy. In all ways but one, the action we use for curved spacetimes is the same as the one just described, albeit with the substitution everywhere of covariant (with respect to diffeomorphisms) derivatives. The only non-minimal change required is that the scalars couple conformally to the Ricci scalar [3]. This ensures the conformal invariance of the theory for general background metrics.

3. Weak-coupling calculation

We now outline the weak-coupling calculation of the effective action and free energy density. Our approach is to compute the contribution of the short-distance, ultraviolet part of the theory to the free energy using the well-established Gilkey-DeWitt heat-kernel methods [15, 16, 17, 18] (see also [19]), which we will briefly review. After the heat kernel discussion we examine what inferences may be drawn from these methods about the long-distance contributions to the free energy.

3.1 Heat-kernel methods

Heat-Kernel techniques are based upon the following representation of the functional determinants which appear at one loop. Assuming euclidean signature in an n-

dimensional spacetime [15, 17, 18],

$$\begin{aligned}\Sigma &= \pm\frac{1}{2}\,\mathrm{Tr}\log\Big(-\Box + m^2 + X\Big)\\ &= \mp\frac{1}{2}\int dV \int_0^\infty \frac{ds}{s}\;\; \mathrm{tr}\;\; K(x,x;s)\,,\end{aligned} \tag{3.1}$$

where the upper (lower) sign is for bosons (fermions), dV is the covariant volume element, and $K(x,y;s)$ satisfies the equation:

$$\frac{\partial K}{\partial s} + \Big(-\Box + m^2 + X\Big)\; K = 0\,, \tag{3.2}$$

with initial condition $K(x,y;s) \to \delta^n(x,y)$ as $s \to 0$.

Part of the utility of this representation lies in the observation that the contribution of the short-distance part of the system to Σ is controlled by the small-s behaviour of $K(x,y;s)$. Furthermore, this small-s behaviour has been computed once and for all, for general choices for the operator $-\Box + X$.

Concretely, if $K(x,y;s)$ is expanded for small s:

$$K(x,y;s) = K_0(x,y;s)\; \sum_{k=0}^\infty a_k(x,y)\; s^k\,, \tag{3.3}$$

with $K_0(x,y;s)$ an appropriately-chosen function [18], then Σ takes the following form:

$$\Sigma = \mp\,\frac{1}{2\,(4\pi)^{n/2}}\; \sum_{k=0}^\infty c_k\; \int dV\;\; \mathrm{tr}\;\; a_k(x,x)\,, \tag{3.4}$$

where c_k represent the following integrals:

$$\begin{aligned}c_k &= \int_0^\infty \frac{ds}{s}\;\; s^{k-n/2}\;\; \exp(-m^2 s)\\ &= m^{n-2k}\,\Gamma\left(k - \frac{n}{2}\right),\end{aligned} \tag{3.5}$$

and the first few $a_k(x,x)$ are given explicitly by [16, 18]:[1]

$$\begin{aligned}a_0(x,x) &= I\,,\\ a_1(x,x) &= -\,\frac{1}{6}\,\Big(R + 6X\Big)\,,\\ a_2(x,x) &= \frac{1}{360}\,\Big(2\,R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + 5R^2 - 12\,\Box\, R\Big)\; I\, +\\ &\quad + \frac{1}{6}\; R\, X + \frac{1}{2}\; X^2 - \frac{1}{6}\,\Box X + \frac{1}{12}\; Y_{\mu\nu}Y^{\mu\nu}\,.\end{aligned} \tag{3.6}$$

[1]In this section, we adopt the curvature conventions of ref. [20].

In this last expression $Y_{\mu\nu}$ is defined by $Y_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$, where ∇_μ is the gauge and coordinate covariant derivative appearing in $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. In the presence of background gravitational and gauge fields, this commutator becomes:

$$Y_{\mu\nu} = -iF^a_{\mu\nu}\, t_a - \frac{i}{2}\, R_{\mu\nu}{}^{\alpha\beta}\, J_{\alpha\beta}\,, \tag{3.7}$$

where t_a and $J_{\alpha\beta}$ represent the generators of gauge and Lorentz transformations on the field of interest.

3.2 Applications to spins zero, half and one

We next record the above expressions for the three spins of interest in the present problem. For the moment we leave the dimension of spacetime arbitrary, although — with dimensional regularization in mind — we imagine evaluating $n = 4 - 2\varepsilon$ at the end of the calculation. Our results are given in the presence of background metric and gauge fields. (Background scalars are also implicitly included through the dependence on the particle mass, m.)

Spin zero. For spinless particles we consider the operator $-\Box + m_0^2 + \xi\, R$ acting on real scalar fields, where the choices $\xi = 0$ and $\xi = -1/6$ respectively correspond (in four dimensions) to a minimally-coupled and conformally-coupled scalar. Denoting the gauge generators as represented on scalars by t_a, and assuming that we have N_0 scalars sharing the same value of ξ we find:

$$Y_{\mu\nu} = -i\, F^a_{\mu\nu}\, t_a\,, \tag{3.8}$$

and so the $a_k(x,x)$ become:

$$\begin{aligned} \mathrm{tr}_0\, a_0(x,x) &= N_0\,, \\ \mathrm{tr}_0\, a_1(x,x) &= -N_0\, \left(\xi + \frac{1}{6}\right)\, R\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \mathrm{tr}_0\, a_2(x,x) &= N_0 \left[\frac{1}{180}\, R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - \frac{1}{180}\, R_{\mu\nu}R^{\mu\nu} + \frac{1}{2}\, \left(\xi + \frac{1}{6}\right)^2 R^2 - \right. \\ &\qquad \left. - \frac{1}{6}\, \left(\xi + \frac{1}{5}\right)\, \Box\, R\right] - \frac{1}{12}\, C(\mathcal{R}_0)\, F^a_{\mu\nu}F_a^{\mu\nu}\,. \end{aligned} \tag{3.10}$$

Here, and in what follows, $\mathcal{R}_s$ denotes the gauge representation of the particles of spin s, and the Dynkin index, $C(\mathcal{R})$, is defined by $\mathrm{tr}(t_a t_b) = C(\mathcal{R})\, \delta_{ab}$. We normalize the gauge generators so that $C(F) = 1/2$ for the fundamental representation of $\mathrm{SU}(N_c)$.

Spin one half. Without loss of generality we represent spin-half particles using Majorana spinors. For such fields the operator of interest is $\not{\nabla} + m_{1/2}$, for which the determinant is not well-defined. In the absence of anomalies in the Lorentz or gauge group, we instead define this determinant as the square root of the determinant of $\left(-\nabla\!\!\!\!/ + m_{1/2}\right)\left(\nabla\!\!\!\!/ + m_{1/2}\right)$, implying

$$\begin{aligned} \mathrm{Tr}\log\left(\nabla\!\!\!\!/ + m_{1/2}\right) &= \frac{1}{2}\ \mathrm{Tr}\log\left(-\nabla\!\!\!\!/^2 + m_{1/2}^2\right) \\ &= \frac{1}{2}\,\mathrm{Tr}\log\left[-\Box + m_{1/2}^2 - \frac{1}{4}\,R + \frac{i}{2}\,\gamma^{\mu\nu}F^a_{\mu\nu}\,\left(T_a\gamma_L - T^*_a\gamma_R\right)\right]. \end{aligned} \tag{3.11}$$

Here $\gamma_{\mu\nu} = \frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right]$ and T_a is the gauge generator as represented on left-handed fermions.

The previous formalism may now be applied to this expression, provided the extra factor of '1/2' seen in eq. (3.11) is kept in mind. The traced commutator in this case is:

$$\mathrm{tr}\ \left(Y_{\mu\nu}Y^{\mu\nu}\right) = -\frac{1}{2}\,N_{1/2}\,R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4C\left(\mathcal{R}_{\frac{1}{2}}\right)F^a_{\mu\nu}F_a^{\mu\nu}\,, \tag{3.12}$$

and so the $a_k(x,x)$ become:

$$\mathrm{tr}_{\,1/2}\ a_0(x,x) = 2N_{1/2}\,,$$

$$\mathrm{tr}_{\,1/2}\ a_1(x,x) = \frac{1}{6}\,N_{1/2}\,R, \tag{3.13}$$

$$\begin{aligned} \mathrm{tr}_{\,1/2}\ a_2(x,x) &= N_{1/2}\left[-\frac{7}{720}\,R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - \frac{1}{90}\,R_{\mu\nu}R^{\mu\nu} + \frac{1}{144}\,R^2 + \frac{1}{60}\,\Box R\right] + \\ &\quad + \frac{1}{3}\,C\left(\mathcal{R}_{1/2}\right)\ F^a_{\mu\nu}F_a^{\mu\nu}\,. \end{aligned} \tag{3.14}$$

Again $N_{1/2}$ denotes the total number of Majorana spinors involved.

Spin one. Gauge potentials, A^a_μ, are the fields representing spin-one particles. Their contribution to the one-loop effective action takes a form for which the above-described formalism applies provided that the gauge is chosen appropriately. For background-covariant Feynman gauge the appropriate differential operator for A^a_μ has the form $-\,\delta^\mu_\nu\,\delta^a_b\,\Box + \delta^\mu_\nu\,(m_1^2)^a_b + X^{a\mu}_{b\nu}$, with $X^{a\mu}_{b\nu}$ and the traced commutator, $Y_{\mu\nu}$, given in this case by:

$$\begin{aligned} X^{a\mu}_{b\nu} &= -\delta^a_b R^\mu_\nu + 2i\,F^{c\mu}{}_\nu\,(\tau_c)^a_b \\ \mathrm{tr}\left(Y_{\mu\nu}Y^{\mu\nu}\right) &= -N_1 R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + 4C\,(A)\,F^a_{\mu\nu}F_a^{\mu\nu}\,. \end{aligned} \tag{3.15}$$

Here $(\tau_a)^b_c = ic^b{}_{ac}$ denotes the gauge generator in the adjoint representation, and $C(A)$ is the corresponding Dynkin index. (For $\mathrm{SU}(N_c)$ we have $C(A) = N_c$.)

In addition to the vector potentials, each spin-one particle also requires a Fadeev-Popov-DeWitt ghost, which contributes like a minimally-coupled complex scalar in the adjoint representation, however it is an anticommuting field. Combining the vector and ghost contributions gives, for N_1 spin-one particles:

$$\begin{aligned}
\mathrm{tr}_1\, a_0(x,x) &= 2N_1\,, \\
\mathrm{tr}_1\, a_1(x,x) &= \frac{2}{3}\, N_1\, R, \qquad (3.16)\\
\mathrm{tr}_1\, a_2(x,x) &= N_1\, \left[-\frac{13}{180}\, R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + \frac{22}{45}\, R_{\mu\nu}R^{\mu\nu} - \frac{5}{36}\, R^2 + \frac{1}{10}\, \Box R\right] + \\
&\quad + \frac{11}{6}\, C\,(A)\ F^a_{\mu\nu}F^{\mu\nu}_a\,. \qquad (3.17)
\end{aligned}$$

3.3 Combining results

We now proceed to the case of interest, $N = 4$ SYM theory in four spacetime dimensions. This corresponds to the choice $N_0 = 6N_1$, $N_{1/2} = 4N_1$ with $N_1 = d_G$ given by the dimension of the gauge group. We also choose all particles to transform in the adjoint representation, $\mathcal{R}_0 = \mathcal{R}_{1/2} = A$, and we take $\xi = -1/6$, as is appropriate for conformally-coupled scalars. For this choice, and combining according to $\mathrm{Tr}\ a_k = \mathrm{tr}_0\, a_k - \mathrm{tr}_{1/2}\, a_k + \mathrm{tr}_1\, a_k$, we find:

$$\begin{aligned}
\mathrm{Tr}\ a_0(x,x) &= N_0 - 2N_{1/2} + 2N_1 = 0\,, \\
\mathrm{Tr}\ a_1(x,x) &= -6\, d_G\left(\xi + \frac{1}{6}\right) R = 0\,, \qquad (3.18)\\
\mathrm{Tr}\ a_2(x,x) &= d_G\left\{\frac{1}{2}\, R_{\mu\nu}R^{\mu\nu} + \left[3\left(\xi+\frac{1}{6}\right)^2 - \frac{1}{6}\right] R^2 - \left(\xi + \frac{1}{6}\right)\Box R\right\} \\
&= \frac{d_G}{2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}\, R^2\right). \qquad (3.19)
\end{aligned}$$

Note that $\mathrm{Tr}\, a_2$ takes the form familiar from the conformal anomaly in which the coefficient of $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$ vanishes [21].[2]

3.4 UV finiteness on ultrastatic spacetimes

As may be seen from the form of the integrals, c_k, of eqs. (3.4) and (3.5), the coefficients $\mathrm{Tr}\, a_k(x,x)$, for $k = 0, 1, 2$ determine the ultraviolet divergences of the four-dimensional theory ($n = 4$). The vanishing of the $k = 0$ and $k = 1$ terms in eq. (3.18) and the fact that the expression for the $k = 2$ term in eq. (3.19) depends only on the background curvature (due to the cancellation among different spins of terms proportional to $F^a_{\mu\nu}F^{\mu\nu}_a$) is a check on the calculation. It shows that this theory, on flat spacetimes, is ultraviolet finite at one loop, as expected. We now show that this cancellation also occurs for *arbitrary* four-dimensional ultrastatic spacetimes.

The finite-temperature calculation of the next section is performed for ultrastatic spacetimes. These admit metrics for which coordinates may be locally chosen to

[2] We thank Arkady Tseytlin for pointing out an error in our original version of this formula.

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ensure $ds^2 = d\tau^2 + \gamma_{mn} dx^m dx^n$, where the spatial metric, γ_{mn}, is independent of τ. Since any such metric is effectively three-dimensional, both its squared Weyl tensor and its Gauss-Bonnet integrand must vanish identically:

$$\begin{aligned} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} &= R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 2\, R_{\mu\nu} R^{\mu\nu} + \frac{1}{3}\, R^2 = 0 \\ \epsilon^{\mu\nu\lambda\rho} \epsilon^{\alpha\beta\gamma\delta} R_{\mu\nu\alpha\beta} R_{\lambda\rho\gamma\delta} &= - R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + 4\, R_{\mu\nu} R^{\mu\nu} - R^2 = 0\,. \end{aligned} \tag{3.20}$$

Eq. (3.20) may be used to eliminate the quantities $R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$ and $R_{\mu\nu} R^{\mu\nu}$ in terms of R^2, with the result:

$$R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = R_{\mu\nu} R^{\mu\nu} = \frac{1}{3}\, R^2\,. \tag{3.21}$$

It is now obvious that eq. (3.19) vanishes when eq. (3.21) is used. Thus we see that $N = 4$ SYM is UV finite at one loop on all four-dimensional ultrastatic spacetimes. This result also extends to background geometries which have a product structure, e.g.,$S^2 \times S^2$. Again, since the individual components of the metric are lower dimensional in such a case, one finds the vanishing of eq. (3.20). However, note that for more general backgrounds, it is quite possible for Tr $a_2(x,x)$ to be nonzero. For instance, for four-dimensional maximally symmetric spaces $R_{\mu\nu\lambda\rho} = \frac{1}{12}\, R\ (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda})$, and so $4 R_{\mu\nu} R^{\mu\nu} = R^2$, giving

$$\mathrm{Tr}\, a_2(x,x) = -\,\frac{d_G}{24}\, R^2 \neq 0\,. \tag{3.22}$$

In such cases logarithmic UV divergences would appear in the effective action.

3.5 Nonzero temperature

The calculation of the ultraviolet part of the one-loop effective action for nonzero temperature is performed along lines very similar to the zero-temperature result just described [22, 19]. We do so by restricting to the ultrastatic metrics of the previous section, and periodically identifying the euclidean time τ with period $\beta = 1/T$. The only change required in the previous calculation is to find the Heat Kernel which is periodic — or, for fermions, antiperiodic — in time.

Such a kernel may be obtained from the one used above at zero temperature through the method of images:

$$K_\pm(t - t', \mathbf{x}, \mathbf{x}'; s) = \sum_{n=-\infty}^{\infty} (\pm)^n K(t - t' + n\beta, \mathbf{x}, \mathbf{x}'; s)\,. \tag{3.23}$$

where, again, the upper (lower) sign is for bosons (fermions). This sum may be evaluated within the small-s expansion for the ultrastatic spacetimes of interest here because the time-dependence of $K_0(\tau - \tau', \mathbf{x}, \mathbf{x}; s)$ is quite simple:

$$K_0(\tau - \tau', \mathbf{x}, \mathbf{x}; s) = K_0(\mathbf{x}, \mathbf{x}; s)\ \exp\left[-\,\frac{(\tau - \tau')^2}{4s} \right]. \tag{3.24}$$

Expanding $K_\pm$ for small s, as in eq. (3.3), and using the known form for K_0, one finds [22]:

$$K_\pm(x,x;s) = K_0(\mathbf{x},\mathbf{x};s) \sum_{n=-\infty}^{\infty} (\pm)^n \exp\left(- \frac{n^2\beta^2}{4s}\right) . \tag{3.25}$$

The sums may be performed, giving standard Jacobi ϑ-functions [23]:

$$K_+(x,x;s) = \vartheta_3(\tau)\; K_0(\mathbf{x},\mathbf{x};s)\,, \qquad K_-(x,x;s) = \vartheta_4(\tau)\; K_0(\mathbf{x},\mathbf{x};s)\,, \tag{3.26}$$

where $\tau = i\beta^2/(4\pi s)$.

Substituting eq. (3.26) into expression (3.1) for the effective action, one then finds that the free energy density is:

$$F = \mp \frac{1}{2\,(4\pi)^{n/2}} \sum_{k=0}^{\infty} C_k^{(\pm)} \;\mathrm{tr}\; a_k(x,x)\,, \tag{3.27}$$

where now $C_k^{(\pm)}$ represent the following integrals:

$$\begin{aligned} C_k^{(+)} &= \int_0^\infty \frac{ds}{s}\; s^{k-n/2}\; \exp(-m^2 s)\; \vartheta_3\left(\frac{i\beta^2}{4\pi s}\right), \\ C_k^{(-)} &= \int_0^\infty \frac{ds}{s}\; s^{k-n/2}\; \exp(-m^2 s)\; \vartheta_4\left(\frac{i\beta^2}{4\pi s}\right). \end{aligned} \tag{3.28}$$

These integrals diverge for $k = 0, 1$ and 2 if $n = 4$ due to the limit of integration at $s \to 0$. This infinity is an ultraviolet divergence, and is removed if one focuses purely on the temperature-dependent part of the problem. This is accomplished by computing $\Delta_k^{(\pm)} = C_k^{(\pm)} - c_k$, which gives the change of the free energy relative to the zero-temperature result: $\Delta F = F(T) - F(0)$.

The potential infrared divergence as $s \to \infty$ does not arise so long as m is nonzero, so $\Delta_k^{(\pm)}$ may be evaluated numerically to any desired accuracy. For our later purposes, however, it is instructive to examine the high-temperature limit, $m \ll T$. This limit can be subtle due to the shadow cast by incipient infrared divergences which can re-emerge in this regime. A well-known example of this occurs in flat space, since $\Delta_0^{(+)}$ acquires terms such as $m^3 T$ when expanded in powers of m/T [24], [25]. This nonanalytic dependence of $\Delta_0^{(+)}$ on m^2 as $m^2 \to 0$ thwarts its evaluation *via* term-by-term integration after expanding its integrand in powers of m^2, since the successive integrals which are obtained in this way diverge more and more severely in the infrared. Such nonanalytic dependence on m^2 is characteristic of the singularities which arise in the presence of massless particles.

For nontrivial background fields we see that evaluating at $m = 0$ causes the $\Delta_k^{(\pm)}$ to diverge at $s \to \infty$ for sufficiently large k. This is very similar to the divergences which are found when a series in powers of m^2 is attempted in the absence of background fields. In this case the divergences are a consequence of a breakdown of the derivative expansion due to the nonanalytic dependence on the background fields which is generated as the curvatures cut off the infrared singularities which are generated by the various massless modes [26].

In four dimensions these divergences potentially arise at $m = 0$ when $k \geq 2$. We now explicitly display the first of these by evaluating the terms $k \leq 2$ of the Gilkey-De Witt expansion. One finds a result expressed in terms of Euler's Γ-function and Riemann's ζ-function:

$$\begin{aligned}\Delta_k^{(+)} &= \int_0^\infty \frac{ds}{s}\; s^{k-n/2}\left[\vartheta_3\left(\frac{i\beta^2}{4\pi s}\right) - 1\right] = 2\left(\frac{4}{\beta^2}\right)^{\frac{n}{2}-k}\Gamma\left(\frac{n}{2}-k\right)\zeta(n-2k)\,,\\ \Delta_k^{(-)} &= \int_0^\infty \frac{ds}{s}\; s^{k-n/2}\left[\vartheta_4\left(\frac{i\beta^2}{4\pi s}\right) - 1\right] = \left(2^{1-n+2k}-1\right)\,\Delta_k^{(+)}\,.\end{aligned} \tag{3.29}$$

Combining all expressions, and expanding about $n = 4$, gives the following result for the free energy density of a general theory:

$$\begin{aligned}\Delta F &= -\frac{1}{2\,(4\pi)^{n/2}}\sum_{k=0}^{\infty}\Delta_k^{(+)}\left[\mathrm{tr}_{\,B} a_k(x,x) + \left(1-2^{1-n+2k}\right)\mathrm{tr}_{\,F} a_k(x,x)\right],\\ &= -\frac{\pi^2 T^4}{90}\left[\mathrm{tr}_{\,B}\; a_0 + \frac{7}{8}\;\mathrm{tr}_{\,F}\; a_0\right] - \frac{T^2}{24}\left[\mathrm{tr}_{\,B}\; a_1 + \frac{1}{2}\;\mathrm{tr}_{\,F}\; a_1\right] - \\ &\quad -\frac{1}{32\pi^2}\left[\frac{2}{4-n} - 2\ln\left(\frac{T}{T_0}\right) + \cdots\right]\left[\mathrm{tr}_{\,B} a_2 - \mathrm{tr}_{\,F} a_2\right] + \frac{3\ln 2}{32\pi^2}\,\mathrm{tr}_{\,F} a_2 + \cdots\,.\end{aligned} \tag{3.30}$$

Here $\mathrm{tr}_{\,B}\; a_k$ and $\mathrm{tr}_{\,F}\; a_k$ respectively denote the trace over the bosons and fermions of the model. The ellipses within the square brackets represent terms which neither diverge, nor involve the logarithm of T. By contrast, the ellipses at the end of the equation represent terms which involve higher than two powers of curvature and gauge field strengths.

The infrared divergence arises in eq. (3.30) as the pole as $n \to 4$ in the $k = 2$ term of the series. Notice that the condition for the cancelling of this particular divergence is $\mathrm{tr}_{\,B}\; a_2 = \mathrm{tr}_{\,F}\; a_2$, which is precisely the condition for the cancelling of the zero temperature logarithmic ultraviolet divergence. As a consequence this particular divergence does not arise in the present example of $N = 4$ SYM, although the same is not expected to be true for the infrared divergences arising for higher k.

Evaluating this expression using the a_k's of $N = 4$ SYM, and using the simplifications, eq. (3.21), which follow from the restriction to an ultrastatic spacetime, finally gives:

$$\Delta F = -\frac{\pi^2 d_G T^4}{6} - \frac{d_G T^2 R}{24} + \frac{d_G \ln 2}{160\pi^2}\Box R + \frac{C(A)\ln 2}{8\pi^2}\;F^a_{\mu\nu}F_a^{\mu\nu} + \cdots\,. \tag{3.31}$$

In the next section we compute the free energy density in the strong-coupling limit using the AdS/CFT correspondence [1]. We do so for the special case of vanishing background gauge fields, and for the specific case where the spatial three-geometry has constant curvature. The Ricci scalar for such a space may be written as

$R = -6\kappa/\ell^2$, where ℓ is the radius of curvature, and $\kappa = +1, 0, -1$ for a three-sphere, flat space and a hyperbolic three-plane. Further for comparison purposes, we will be interested in the gauge group for SU(N_c) in the large-N_c limit, so $d_G = N_c^2 - 1 \approx N_c^2$. For these choices, our result in eq. (3.31) yields:

$$\begin{aligned}\Delta F &= N_c^2 \left[-\frac{\pi^2 T^4}{6} + \frac{\kappa T^2}{4\ell^2} + O\left(\frac{1}{\ell^6 T^2}\right)\right] \\ &= -\frac{\pi^2 N_c^2 T^4}{6}\left[1 - \frac{3\kappa}{2\pi^2 \ell^2 T^2} + O\left(\frac{1}{\ell^6 T^6}\right)\right]. \end{aligned} \tag{3.32}$$

4. Strong-coupling calculation

Using the AdS/CFT correspondence, the dual supergravity description of the $N = 4$ SYM theory at finite temperature is an asymptotically anti-de Sitter black hole [1, 5]. An appropriate class of metrics describing euclideanized black holes is [27]

$$ds^2 = \left(\frac{r^2}{\ell^2} + \kappa - \frac{r_0^2}{r^2}\right) d\tau^2 + \frac{dr^2}{\frac{r^2}{\ell^2} + \kappa - \frac{r_0^2}{r^2}} + \frac{r^2}{\ell^2}\, d\Sigma_3(\kappa)\,, \tag{4.1}$$

where $\kappa = +1, 0, -1$ and $d\Sigma_3(k)$ is the line element on a three-dimensional manifold with constant curvature κ/ℓ^2. An explicit representation of the latter may be chosen as

$$d\Sigma_3(\kappa) = \ell^2 \left[(1 - \kappa\rho^2) dz^2 + \frac{d\rho^2}{1 - \kappa\rho^2} + \rho^2 d\phi^2\right], \tag{4.2}$$

but the precise form of the three-dimensional metric will not be needed in the following. All of these five-dimensional metrics (4.1) satisfy the equation of motion,[3] $R_{\mu\nu} = -(4/\ell^2) g_{\mu\nu}$, which arises from the action

$$I = -\frac{1}{16\pi G_5}\int d^5x \sqrt{g}\left(R + \frac{12}{\ell^2}\right) - \frac{1}{8\pi G_5}\int d^4x \sqrt{h} K\,. \tag{4.3}$$

To avoid a conical singularity at

$$r_+ = \frac{1}{2}\left(\sqrt{\kappa^2 \ell^2 + 4 r_0^2 \ell^2} - \kappa\ell\right), \tag{4.4}$$

where $g_{\tau\tau}$ vanishes, one must select the period of τ to be

$$\frac{1}{T} = \beta = \frac{2\pi \ell^2 r_+}{2 r_+^2 + \kappa\ell^2}\,. \tag{4.5}$$

[3]Note that there is a change of conventions for the curvatures between this section and the previous one. Those of this section follow those of ref. [28], since these are conventional in the gravity literature. Although the metric signature remains $-+++$, we must replace $R_{\mu\nu\lambda\rho} \to -R_{\mu\nu\lambda\rho}$ in all formulae.

Now, the euclidean action evaluated for the classical black hole metric is interpreted as giving the leading contribution to the free energy. In the context of AdS black holes, such calculations were first carried out in ref. [29]. As typically arises in these calculations, however, this action diverges and so to produce a finite result, we subtract off the contribution of a reference metric [30]. This step is analogous to subtracting off the zero-temperature free energy in the field theory calculation — see section 3.5. In this case, the reference geometry is simply anti-de Sitter space which is produced by setting $r_0 = 0$ in eq. (4.1). Taking care to match the asymptotic geometries correctly, one finds

$$\Delta I = V_3 \beta \, \Delta F = V_3 \beta \, \frac{r_+^2 (\ell^2 \kappa - r_+^2)}{16 \pi G_5 \ell^5} \, . \tag{4.6}$$

Note that in defining $\Delta I = V_3 \beta \Delta F$, the relevant volume is essentially the coordinate volume of τ and $d\Sigma_3(\kappa)$. It is not the proper volume of a surface of constant radius at large r in eq. (4.1), which of course diverges as $r \to \infty$.

Now the duality prescription [1] identifies $G_5 \ell^5 = 8\pi^3 g^2 \alpha'^4$ and $\ell^4 = 4\pi g N_c \alpha'^2$, where g and α' are the string coupling constant and the inverse string tension, respectively. Combining these formula then yields the free energy density at strong coupling for SU(N_c) SYM in the large-N_c limit:

$$\Delta F = -\frac{\pi^2 N_c^2 T^4}{8} \mathcal{F}\left(\frac{\kappa \beta^2}{\ell^2}\right), \tag{4.7}$$

where the function $\mathcal{F}(x)$ is given by:

$$\begin{aligned} \mathcal{F}(x) &= \frac{1}{16}\left[1 + \left(1 - \frac{2x}{\pi^2}\right)^{1/2}\right]^2 \left\{\left[1 + \left(1 - \frac{2x}{\pi^2}\right)^{1/2}\right]^2 - \frac{4x}{\pi^2}\right\} \\ &\approx 1 - 3\left(\frac{x}{\pi^2}\right) + \frac{3}{2}\left(\frac{x}{\pi^2}\right)^2 + \frac{1}{4}\left(\frac{x}{\pi^2}\right)^3 + \cdots \end{aligned} \tag{4.8}$$

where the final expansion applies for $x \ll 1$. This expansion corresponds precisely to the high temperature limit in eq. (4.7), which then becomes

$$\begin{aligned} \Delta F &= N_c^2 \left[-\frac{\pi^2 T^4}{8} + \frac{3\kappa T^2}{8\ell^2} - \frac{3\kappa^2}{16\pi^2 \ell^4} + O\left(\frac{1}{\ell^6 T^2}\right)\right] \\ &= -\frac{\pi^2}{8} N_c^2 T^4 \left[1 - \frac{3}{\pi^2}\frac{\kappa}{\ell^2 T^2} + \frac{3}{2\pi^4}\frac{\kappa^2}{\ell^4 T^4} + O\left(\frac{1}{\ell^6 T^6}\right)\right]. \end{aligned} \tag{4.9}$$

The latter is now in a form which is readily compared to the weak coupling result (3.32).

For the $\kappa = 0$ case, of course, the leading term is the full answer up to corrections in the effective coupling. For the $\kappa = +1$ and -1 cases, similar calculations yielding the curvature corrections appear in refs. [12] and [31], respectively.

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Notice that for $\kappa = 0$ or -1, ΔI in eq. (4.6) is always negative, and so the black hole solution (4.1) always provides the dominant saddle-point in the supergravity path integral for any temperature. On the other hand, for $\kappa = +1$, ΔI becomes positive for $r_+ < \ell$, which corresponds to $\beta > 2\pi\ell/3$. Hence in this low temperature range, AdS space itself is the dominant saddle-point. This change at $\beta = 2\pi\ell/3$ is believed to be associated with a deconfinement phase transition in the SYM theory [5].[4] For comparision to the weak-coupling calculation, we are interested in the high temperature regime where the black hole is the dominant AdS saddle-point for all three choices, $\kappa = 0, \pm 1$.

5. Discussion

In comparing the free energy densities calculated at one loop (3.32) and strong coupling (4.7), we see that these expressions, while not identical, yield two expansions which look very similar. Both give a free energy having a high-temperature limit which has the form of a Taylor expansion in powers of β^2/ℓ^2:

$$\Delta F = -\frac{\pi^2 N_c^2 T^4}{6} \sum_{n=0}^{\infty} b_n(\lambda) \left(\frac{2\kappa}{\pi^2 \ell^2 T^2}\right)^n , \tag{5.1}$$

where the coefficients $b_n(\lambda)$ are functions of the effective 't Hooft coupling, $\lambda = g_{\mathrm{YM}}^2 N_c = \ell^4/(2\alpha'^2)$.

The significance of this observation for the strong-coupling expansion is in what eq. (5.1) does *not* contain. In particular, the weak-coupling expansion had the potential to involve both fractional powers of β^2/ℓ^2 (similar to the $m^3 T$ term in $\Delta_0^{(+)}$), as well as terms proportional to $\log\beta$ (such as when $\mathrm{Tr}\, a_2 \neq 0$). Such terms reflect the infrared divergences which are associated with the massless bosons of the perturbative spectrum. They do not arise at one loop in the weak-coupling calculation due to the cancellations in $\mathrm{Tr}\, a_2$, which reflect the finiteness and conformal invariance of $N = 4$ SYM theory in ultrastatic background geometries. This same cancellation does not hold in general for all terms in the weak-coupling limit. For instance, even on flat space the massless gauge bosons generate infrared divergences to perturbative calculations of thermal quantities like gluon damping rates [32], once one proceeds beyond one loop. The same might also be expected to follow for the $k \geq 3$ terms of the derivative expansion at one loop.

Unlike the weak-coupling result, the strong-coupling expression, eq. (4.8) is not simply given as an asymptotic form for large temperatures, but is explicitly given in terms of the function $\mathcal{F}(\kappa\beta^2/\ell^2)$ for the geometries of interest. Explicit examination of $\mathcal{F}(x)$ clearly shows it to be analytic in its argument near $x = 0$, indicating the

[4] Note that below this phase transition at $\beta = \pi\ell/\sqrt{2}$, there is a branch point in $\mathcal{F}(\beta^2/\ell^2)$. This corresponds to the minimum temperature for which a black hole solution with $\kappa = +1$ exists.

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absence of singularities in the high-temperature regime. But the existence of exact conformal invariance of the underlying model ensures the absence of a gap in the particle spectrum. In four dimensions this follows directly if the ground state respects the conformal invariance, or from the existence of a Goldstone 'dilaton' mode if the conformal symmetry is spontaneously broken. Infrared divergences would be expected to be weakened or absent if the only massless states were Goldstone modes [33], since these decouple in the long-wavelength limit [34]. The absence of infrared singularities in the large temperature limit of the strong-coupling calculation therefore indicates either the existence of only Goldstone massless modes, or the persistence into the large-N_c limit of the one-loop cancellations of infrared divergences which were seen at weak coupling for non-Goldstone massless modes.

Notice there is no singularity predicted by eq. (4.8) for any other temperatures, in spite of the branch point which apparently arises at $x = \pi^2/2$. This branch point has no direct implications because: (i) $x = -\beta^2/\ell^2 \leq 0$ if $\kappa = -1$; (ii) the function $\mathcal{F}(x)$ collapses to a constant (and hence the branch point vanishes) if $\kappa = 0$; and (iii) the branch point lies outside the high temperature domain of validity ($\beta < 2\pi\ell/3$) of the strong-coupling calculation if $\kappa = +1$.

The behaviour we find is consistent with the behaviour which was previously argued for $N = 4$ SYM by Witten [5]. It would be of interest to extend other techniques which have proven powerful in flat space, such as those of ref. [35], to analyse the curved spaces considered here in more detail. In particular, so far as they go our results are consistent with a QCD-like picture in which no phase transition occurs for any finite coupling, λ_c.

Although our calculation cannot reveal the functional form for $b_n(\lambda)$ away from the limits $\lambda \to 0$ and $\lambda \to \infty$, it does differ in some ways from similar calculations of the free energy density in the Higgs phase of the SYM theory [7]. In that case, the results suggest that the strong-to-weak coupling interpolation is achieved with a single overall multiplicative function. While one might have hoped for a similar simple form in the present calculation, it is clear from our expressions that this is not the case. Rather, the weak coupling form (3.32) determines

$$b_0(0) = 1\,, \qquad b_1(0) = \frac{3}{4}\,, \qquad b_2(0) = 0\,, \tag{5.2}$$

while the strong coupling expansion (4.7) fixes

$$b_0(\lambda \to \infty) \to \frac{3}{4}\,, \qquad b_1(\lambda \to \infty) \to \frac{3}{2}\,, \qquad b_2(\lambda \to \infty) \to \frac{3}{8}\,. \tag{5.3}$$

Following the arguments presented in ref. [12], the free energy densities calculated using euclidean techniques above also have an interpretation as Casimir energy densities. The latter arises if the Wick rotation back to a Minkowski signature manifold is made on an appropriate 'time' coordinate in the constant curvature three-manifold.

Then eqs. (3.32) and (4.7) yield the Casimir energy density for SYM on $dS_3 \times S^1$, $R^3 \times S^1$ and $AdS_3 \times S^1$ for $\kappa = +1$, 0 and -1, respectively. The negative Casimir energy is generated by antiperiodic boundary conditions on the S^1 factor for the fermions.

Finally, in closing, we remark that the calculations in section 3 show that in a general curved space, $N = 4$ super-Yang-Mills need not be finite. We observed though that for background geometries with a product structure into two (or more) lower dimensional components, UV finiteness is maintained due to a remarkable cancellation of terms in $\mathrm{Tr}\, a_2$, which is given in eq. (3.19). Indeed, in the finite temperature calculations, finiteness arose because of the ultrastatic form of the background metric, which gives a product structure of the form $S^1 \times M_3$. Of course, it is not difficult to find backgrounds where no such cancellation occurs. For example, consider the simple case of S^4, which is maximally symmetric with $R = 12/\ell^2$ where ℓ is the radius of curvature. In this case eq. (3.22) implies the nonzero result

$$\mathrm{Tr}\ a_2 = -\, 6\, \frac{N_c^2}{\ell^4}\,, \tag{5.4}$$

and hence a logarithmic UV divergence appears in the SYM effective action.

It is not difficult to choose coordinates on five-dimensional AdS space such that the ‘boundary’ manifold (i.e. the asymptotic regulator surfaces) is S^4, i.e.

$$ds^2 = \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2 d\Omega_4\,, \tag{5.5}$$

where $d\Omega_4$ is the standard metric on a unit four-sphere. Now the logarithmic UV divergence appearing at one-loop in the SYM effective action would at first sight seem to present a problem for the proposed AdS/CFT correspondence, but in fact, it provides a remarkable consistency check [36]. The AdS/CFT correspondence provides a new set of intrinsic surface terms [37, 36] (for related work, see [38]) for the gravitational action (4.3). Remarkably, these new terms render the gravitational action finite except precisely in the case where $\mathrm{Tr}\, a_2 \neq 0$ for the boundary surface. In the latter case, the volume integral produces a logarithmic divergence instead. For the S^4 boundary, one can precisely match this divergence with that appearing in the dual field theory calculation [36]. Hence, in keeping with the UV/IR relation [39] of the AdS/CFT correspondence, one again finds that an ultraviolet divergence in the field theory is matched by an infrared divergence in the supergravity.

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